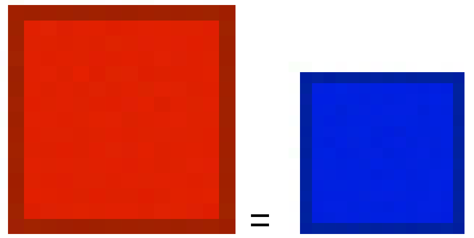
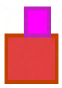
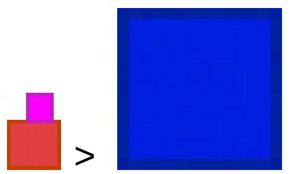
Rectangle Arithmetic

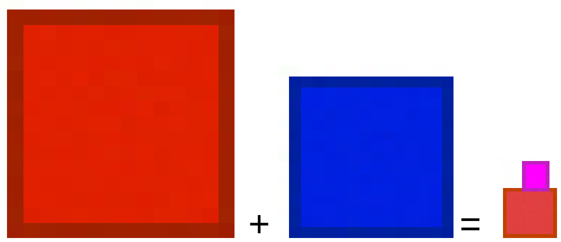
Another slant on fractions*

Represent numbers by boxes. A square, , regardless of size, has value 1:

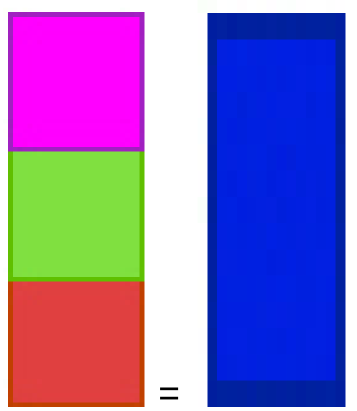


A stack of two squares, , means 2:

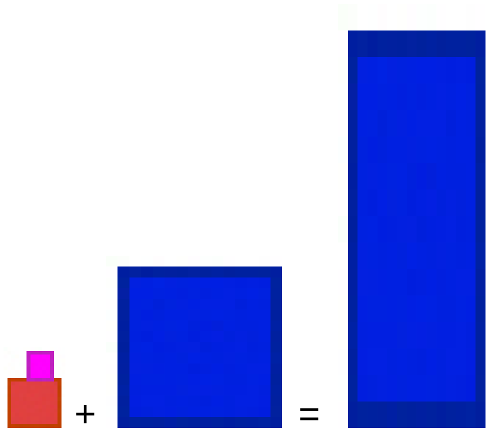




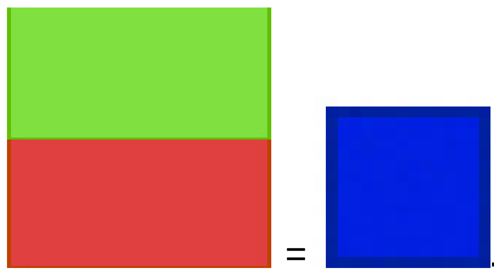
A stack of equal squares can form a rectangle:



This means 1+1+1 = 3. So,

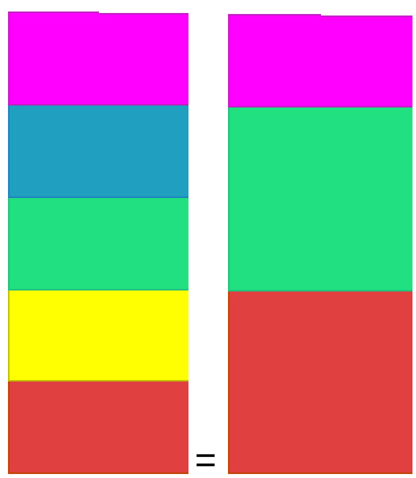


Size doesn't matter--only shape. We add rectangles by stacking them vertically, like with squares:



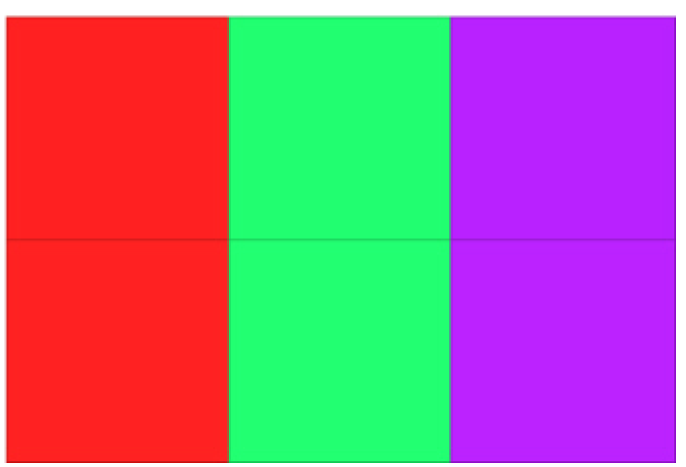
Here, two equal rectangles sum to a "1". *I.e., x+x*=1. So the rectangles are each x = 1/2 = "2" turned sideways.

A tall rectangle that is not a whole number of squares is an "improper" fraction. By marking off the squares, we make a mixed number:

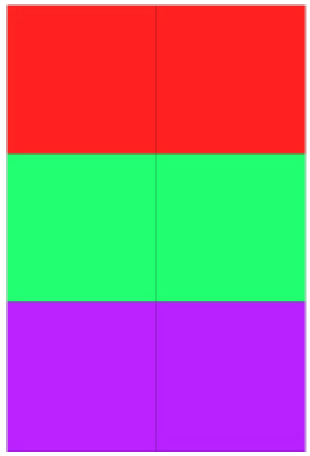


I.e., 5/2 = 2+1/2.

A "3" turned sideways is obviously 1/3. And 2/3 is



which is 3/2,

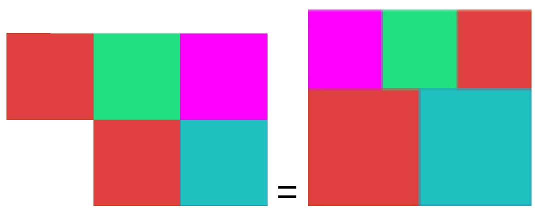
 ,

turned sideways.

Turning any rectangle sideways reciprocates its value, which is just its height divided by its width! That is the slope of its diagonal. Engineers use the fancy term "aspect ratio," but they like to make it ≥ 1 by sometimes switching height and width. If we did that, we'd confuse 3 with 1/3!

So rectangles just represent fractions--the height is the numerator; the width is the denominator. The reason size doesn't matter is that magnifying a rectangle is the same as multiplying the numerator and denominator by the same quantity.

When adding 1/3+1/2, scaling the summands to have the same width and make a nice rectangle is the same as finding a common denominator.

 .

This equation, and the whole idea that shape matters but not size, may seem artificial and slapdash, but there is actually a simple physical example of this behavior. If each square is made of the same electrically resistive material, and we coat their top and bottom edges with a good conductor, and then apply a voltage between the topmost and bottommost edges of the above figures, the currents they pass will be equal, and will not change if the figures are scaled up or down. To help your intuition, suppose you have a square conducting a certain current. Placing another beside it (creating the rectangle value "1/2")

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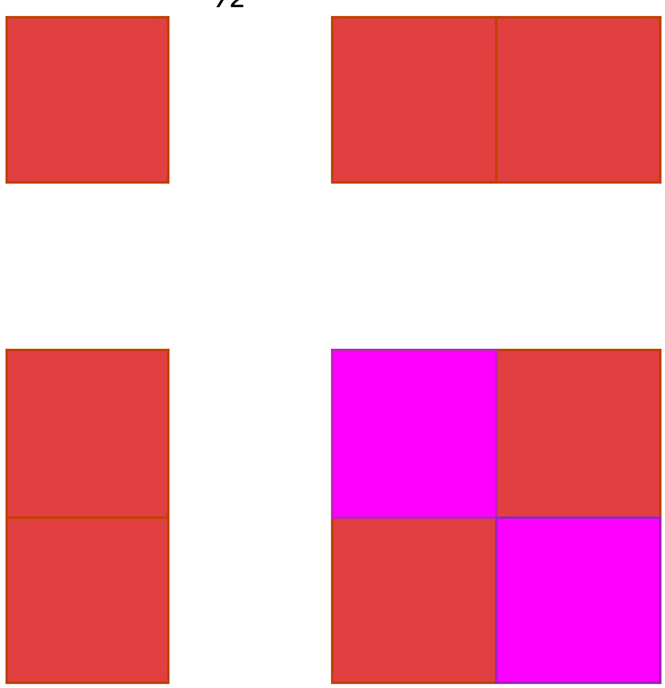
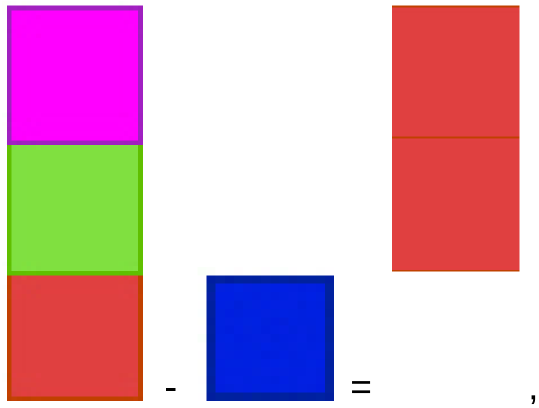


2 1

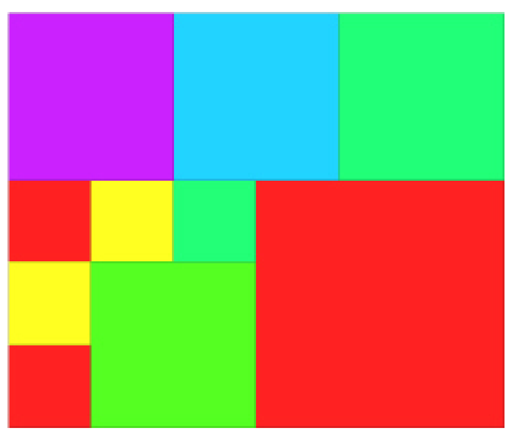
doubles the current, and thus halves the resistance. But then stacking two of these rectangles vertically creates a large square, and redoubles the resistance back to the initial value.

We can subtract a smaller rectangle from a larger one by scaling to equal width (finding a common denominator), and lopping off the smaller from the larger:

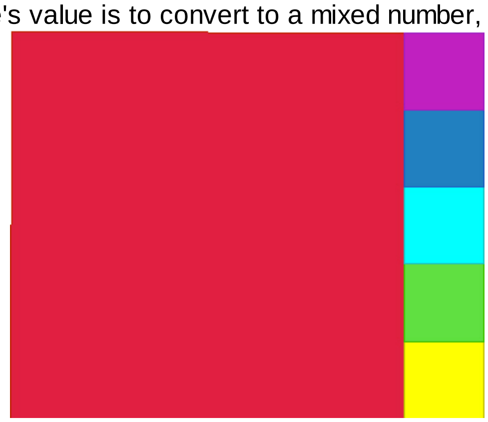


but the extra machinery we'd need for handling negative numbers probably isn't worth it: Draw each rectangle's diagonal; the opposite diagonal means the opposite sign; vertical flipping reverses the sign; scale to equal width before adding; never let diagonals join end to end; instead, superpose the rectangles to have a common upper or lower edge; the sum is the rectangle whose diagonal joins the beginning of one diagonal with the end of the other.

From the 1/2+1/3 example we can read off the answer 5/6, i.e. 5 high and 6 wide, if we subdivide one of the larger squares by trisecting its edges:



A more methodical way to read off a rectangle's value is to convert to a mixed number, reciprocate the fraction, and repeat:

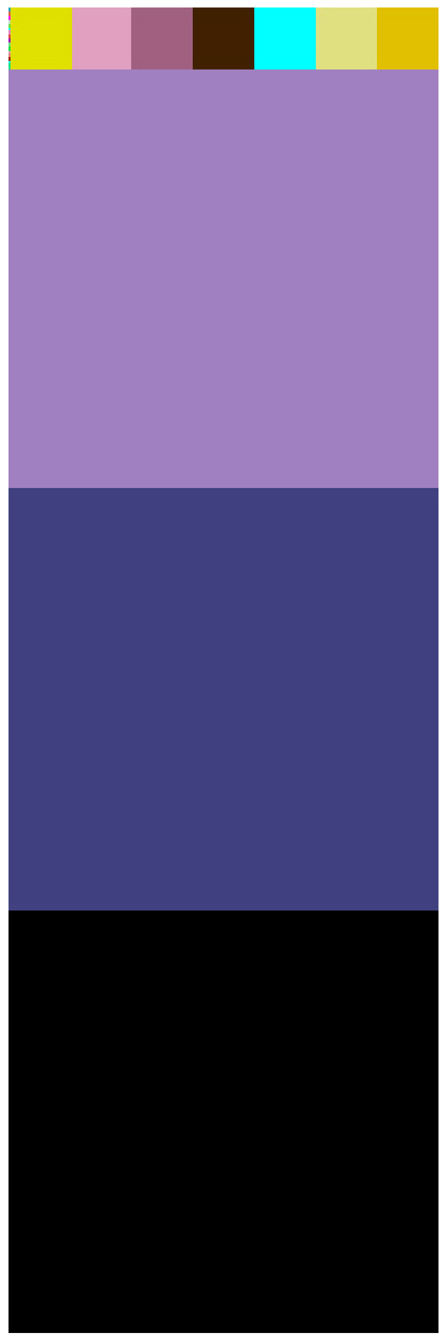


We removed zero squares in the vertical dimension, because the fraction was "proper". Then we got one square horizontally. Then we got five vertically with no remainder, so the process (known as a continued fraction) terminated with the value

$$0+\cfrac{1}{1+\cfrac{1}{5}}=\frac{5}{6}$$

When the slope is not a rational fraction, the continued fraction process does not terminate, as with the number

$$\pi=3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\cfrac{1}{\ddots}}}}}:$$



Due to limited resolution, the column of 15 (more nearly 16) is barely visible. We call the sides of such a rectangle "incommensurable" because there is no scale of measurement in which both are whole numbers. Scaling a rectangle doesn't affect the commensurability of its sides.

What does it mean to stack rectangles horizontally rather than vertically? *I.e.*, what is the slope of a rectangle joined by stacking equally tall rectangles horizontally? Easy: If we turn it sideways, it's the sum of the reciprocals. Reciprocating the sum of the reciprocals (**harmonic sum**) is also how you add resistors in parallel. If we add two frequencies or angular velocities, we harmonically sum the periods (and wavelengths). For example, the time it takes the stars to fully circle Polaris is

$$1 \text{ sidereal day} = \frac{1}{\dfrac{1}{1 \text{ year}} + \dfrac{1}{1 \text{ day}}}$$

because the Earth both rotates and revolves.

Algebra exercise: This rectangle is composed of nine unequal squares. Is it a perfect square?

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If not 1, what number does it represent? Hint: Arbitrarily assign the value 1 to the sides of the little red square, and the value x to the adjoining blue square. Then their green neighbor on the left has side x+1. You can continue assigning sizes in terms of x to all nine squares. (Notice that the central, chartreuse square is "yellow plus magenta" minus "blue plus green". This will equal 4, regardless of x.) Now, equate the top edge of the rectangle with the bottom, and you should get an equation that determines x. Answer: $x = 7$, and the rectangle has slope 32/33. For a dozen or so more of these diagrams, click "Rectangles cut into (mostly) unequal squares" at www.tweedledum.com/rwg/. For hundreds more (with the sizes filled in), search the Web for "squared rectangles". If I were king, one of these diagrams (undimensioned) would appear daily in the newspaper puzzle pages, along with Yesterday's Answer. Sundays would feature an extra large one requring simultaneous equations.

Martin Gardner's "The 2nd Scientific American Book of Mathematical Puzzles and Diversions", Simon and Schuster, 1961, covered the ancient problem of finding a squared (or "perfect") *square*, first solved in the late 1930s.

Besides adding and subtracting, it's easy to **multiply** slopes: If four rectangles fit to form a larger one thus,

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then the products of the diagonally opposite slopes are equal. In this case, blue × olive = red × purple. This gives the term **cross product** a whole old meaning. If the blue (really cyan) is square, then this says 2 × 1/3 = 2/3, (red × purple = olive). Making the red square instead, we have 1/3 × 1/2 = 1/6, (olive × cyan = purple):

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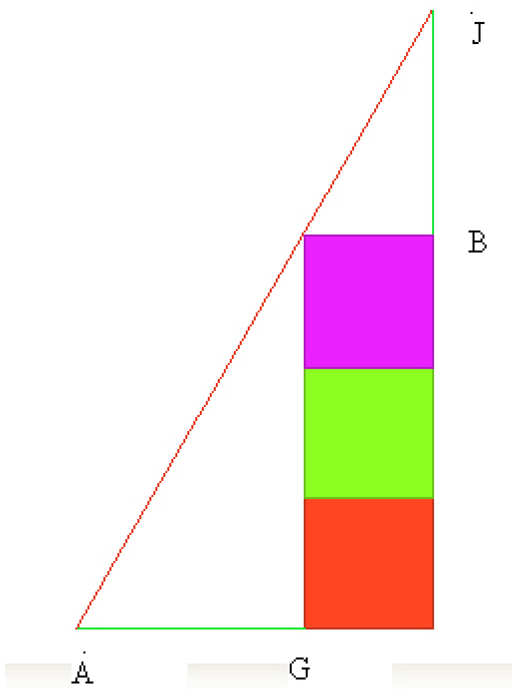
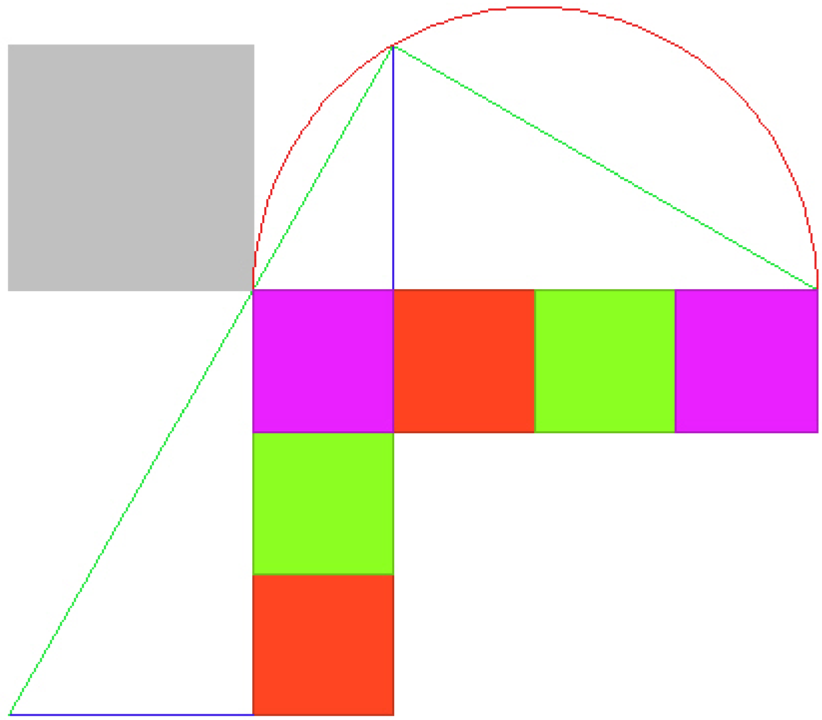
For **division**, just put the dividend diagonally opposite the "1" square, or reciprocate the multiplier.

Note that **we can represent the number 0 = 0/1 = 0/2 = ... with a horizontal stroke, _, *i.e.,* a box of zero height**. You can safely reciprocate it to make 1/0 = 2/0 = ..., represented by a harmless vertical stroke, |. We see that stacking up zeroes makes no difference, and multiplying by "|" makes "|", except that multiplying it by "_" makes a single point, "." = 0/0, and trying to do anything with this just makes another ".".

Here is how to take the **square root** of a slope. Let B stand for Bond. James Bond. He and Auric Goldfinger, G, creep out from opposite corners of the rectangular quarters of *M*. Slope *M*. They creep at identical speeds, so that JB = AG. As soon as they can see each other, they shoot. The line of fire JA has slope $\sqrt{M}$.

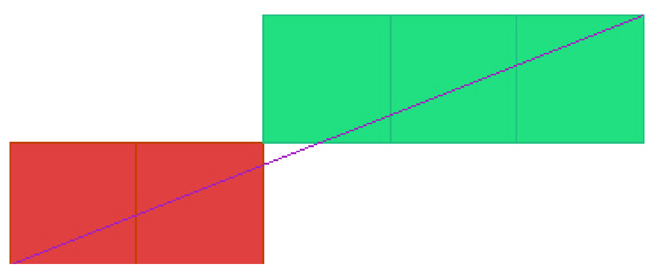


In this illustration, M=3, creating half an equilateral triangle. But how do we actually synchronize Bond's motion with Goldfinger's, and find the line JA? Without scaling, adjoin to the rectangle a sideways copy, thus:



Join the two upper corners of the combined figure with a semicircle. This intersects Bond's path at the desired point J, from which we draw straight lines (green) through those upper corners. The lines will be perpendicular, with the desired (reciprocal) slopes $s = \sqrt{M}$ and $-1/\sqrt{M}$, as shown by the gray square (slope 1) and the multiplication rule, $1 \times M = s \times s$. In fact, we don't even need the gray square to see this: instead use the magenta elbow square as the lower left pane of the cross product $1 \times (1/s) = s \times (1/M)$ (ignoring signs). (To see that the green lines are perpendicular, draw a radius to J, making two isosceles triangles with supplementary apex angles whose average coincides with the angle at J.)

One last operation on rectangles is to join them corner-to-corner:



and then take the bounding box. This example says **mediant**(1/2,1/3) = 2/5. This is the fraction between 1/2 and 1/3 which has the smallest numerator and denominator. In general, the mediant is the sum of the numerators over the sum of the denominators--the way you're **not** supposed to add fractions in grade school. You can find the best rational approximations to any number between 0 and infinity by repeatedly taking the mediant of the last underestimate and overestimate, starting with an underestimate _ (0) and an overestimate |, (infinity). *E.g.*, what's the easiest way to bat .239? Write the underestimates on the left and the overestimates on the right, working toward the middle:

$\frac{0}{1}$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\frac{1}{0}$

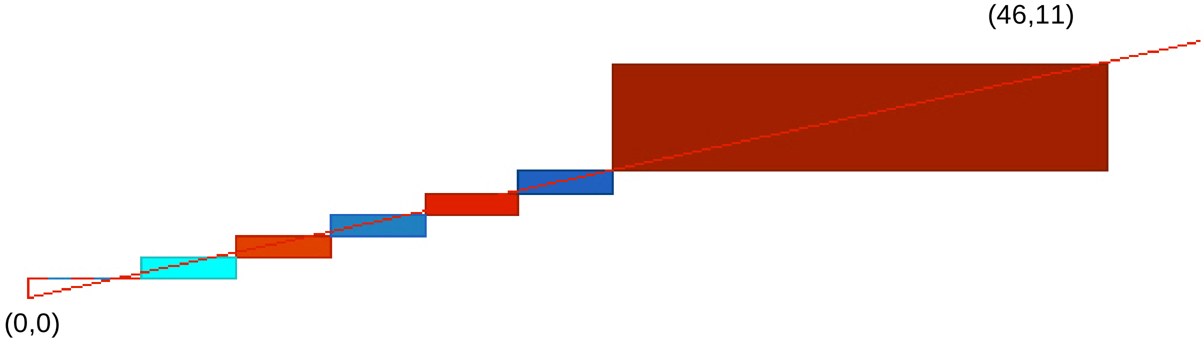
$\frac{1}{5}$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\frac{1}{4}$ $\frac{1}{3}$ $\frac{1}{2}$ $\frac{1}{1}$

$\frac{2}{9}$ $\frac{3}{13}$ $\frac{4}{17}$ $\frac{5}{21}$ $\qquad\qquad$ $\frac{6}{25}$

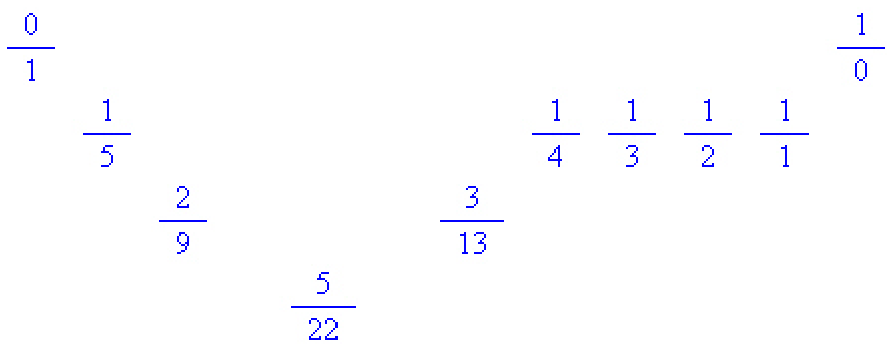
$\frac{11}{46}$

(5/21 ≈ .2381, and 11/46 ≈ .2391 .) So you need (at least) 46 at-bats to bat .239. Graphically, a slope of exactly .239 requires a rectangle with one corner at (0,0) and the opposite at (1000,239), way off the page. We chain together a | (1/0) putting us too high

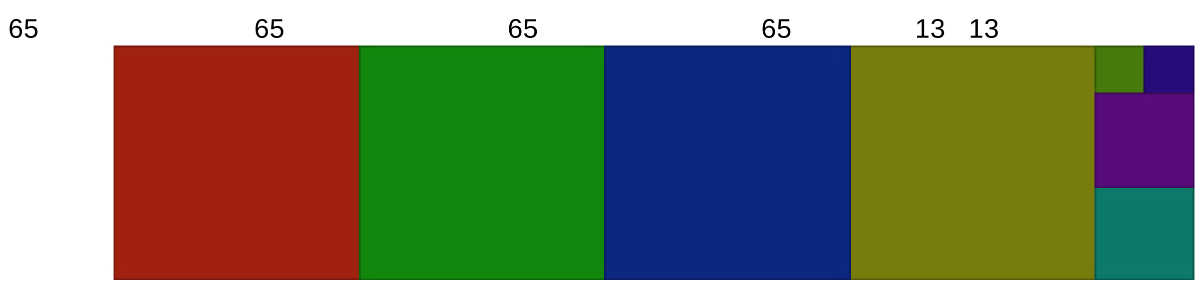
Then five _s (0/1) put us at (5,1), too low. Then five 1/4s put us a bit too high, and finally, a 5/21 gets us within .0005:



The big brown rectangle is the same as the box bounding (0,0) and both red rectangles. If we continue the process, we will soon reach the point (1000,239), which is the first time the slope will be exactly right, since 239/1000 is in lowest terms. The gridpoint nearest (0,0) through which a diagonal passes is its slope in lowest terms. This mediant process finds that point, and thus reduces fractions without first finding the greatest common divisor. E.g., for 65/286 ≈ .2273



where the 5/22 is in the middle because it's exact. The "usual" way to do this graphically is to repeatedly eat off the largest square from a 65 by 286 rectangle, which will terminate with the removal of a 13 by 13 square, which is the greatest common divisor:



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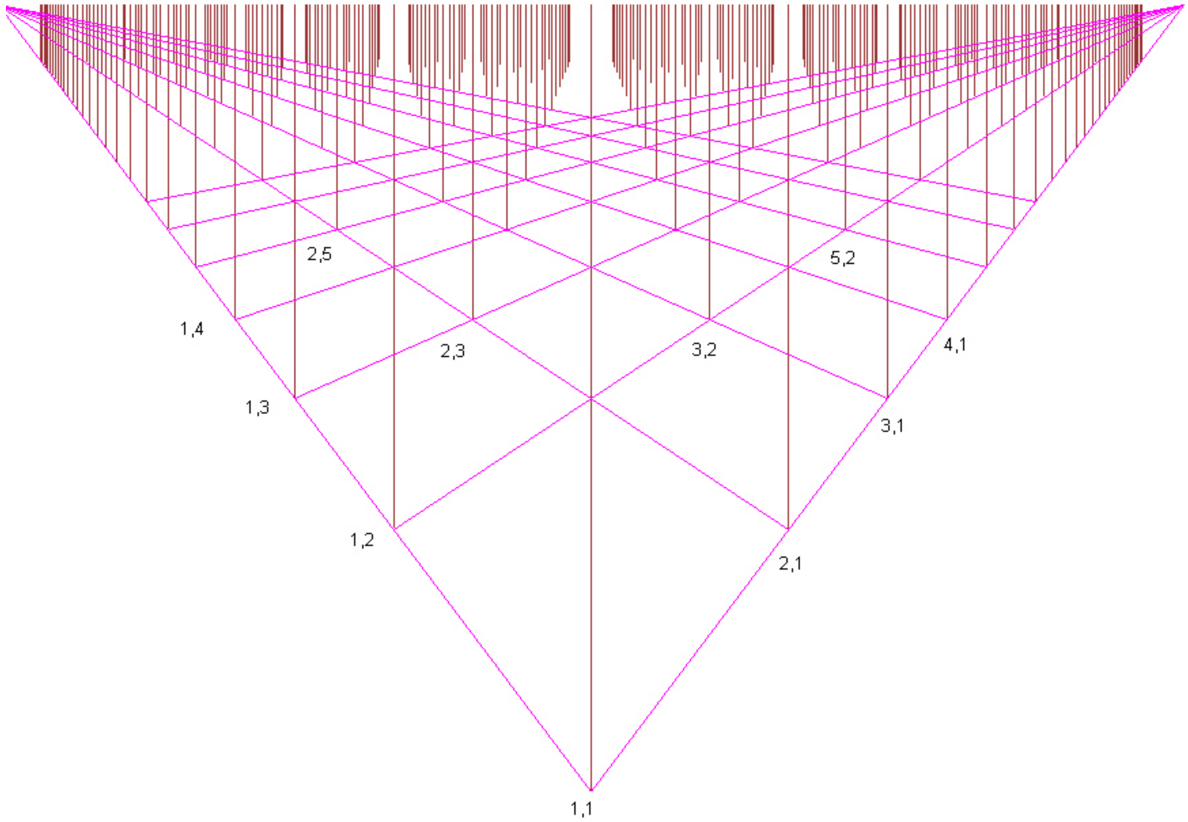
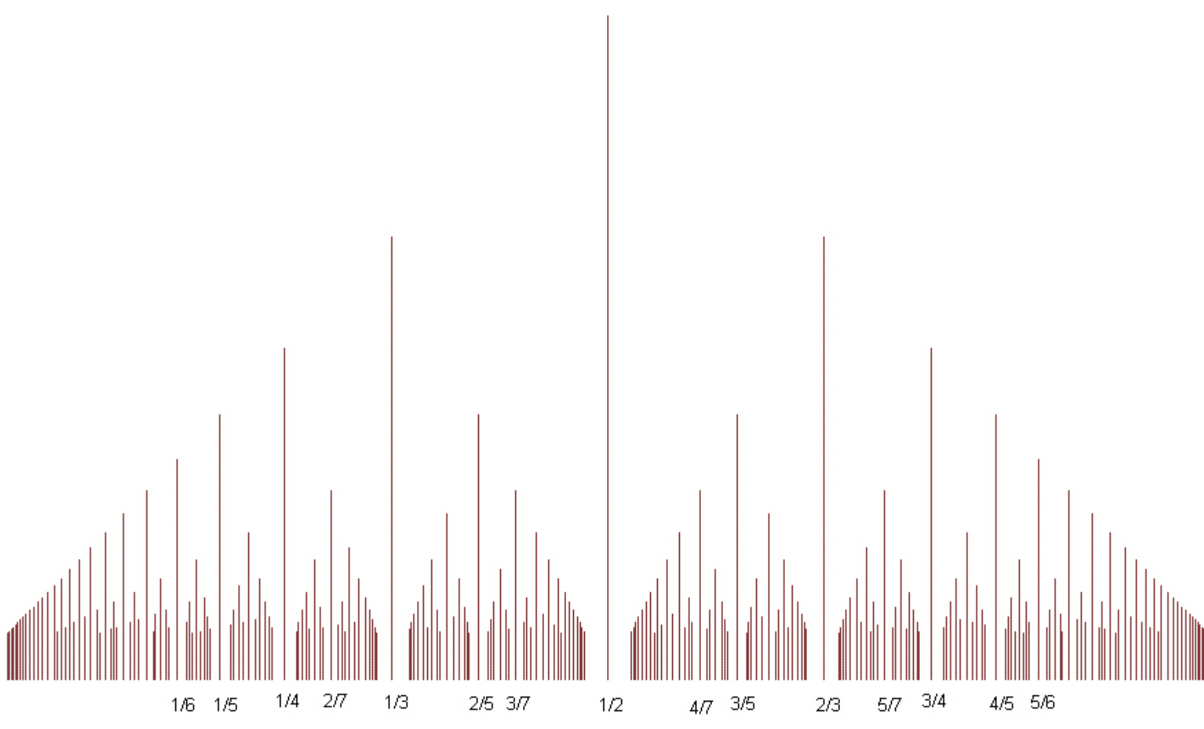
which you then divide out.

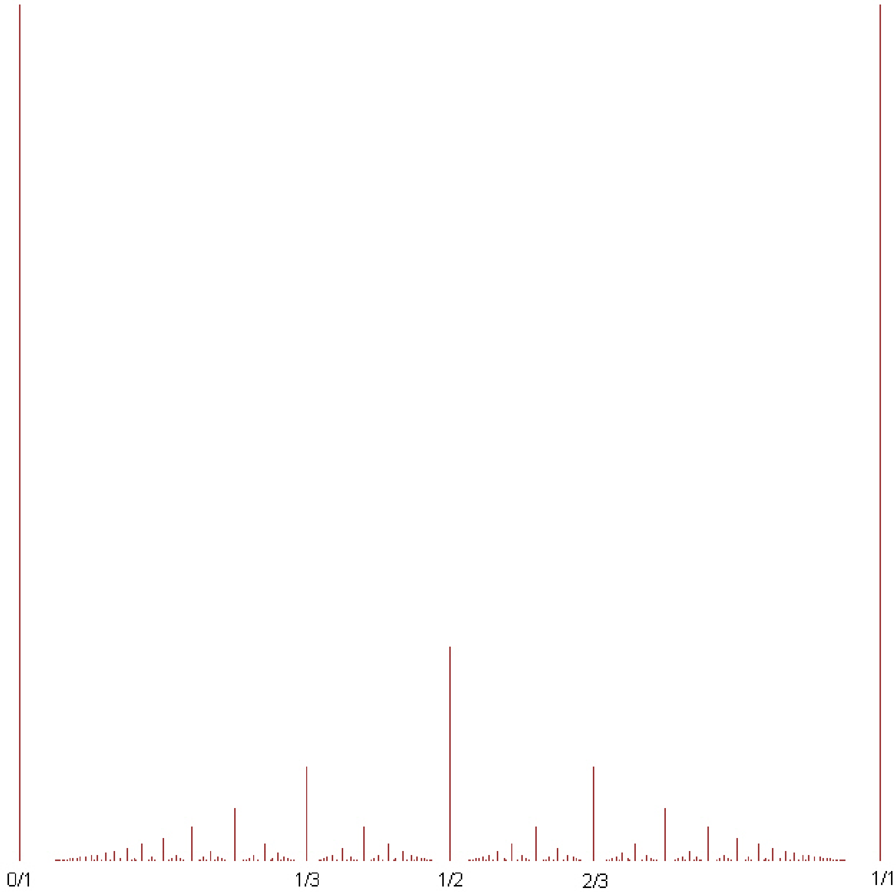
Instead of rectangles, it is also possible to compute mediants with circles. Here is a perspective view of "Euclid's orchard" as described in the Lattice of Integers chapter of Martin Gardner's Sixth Book of Mathematical Diversions.



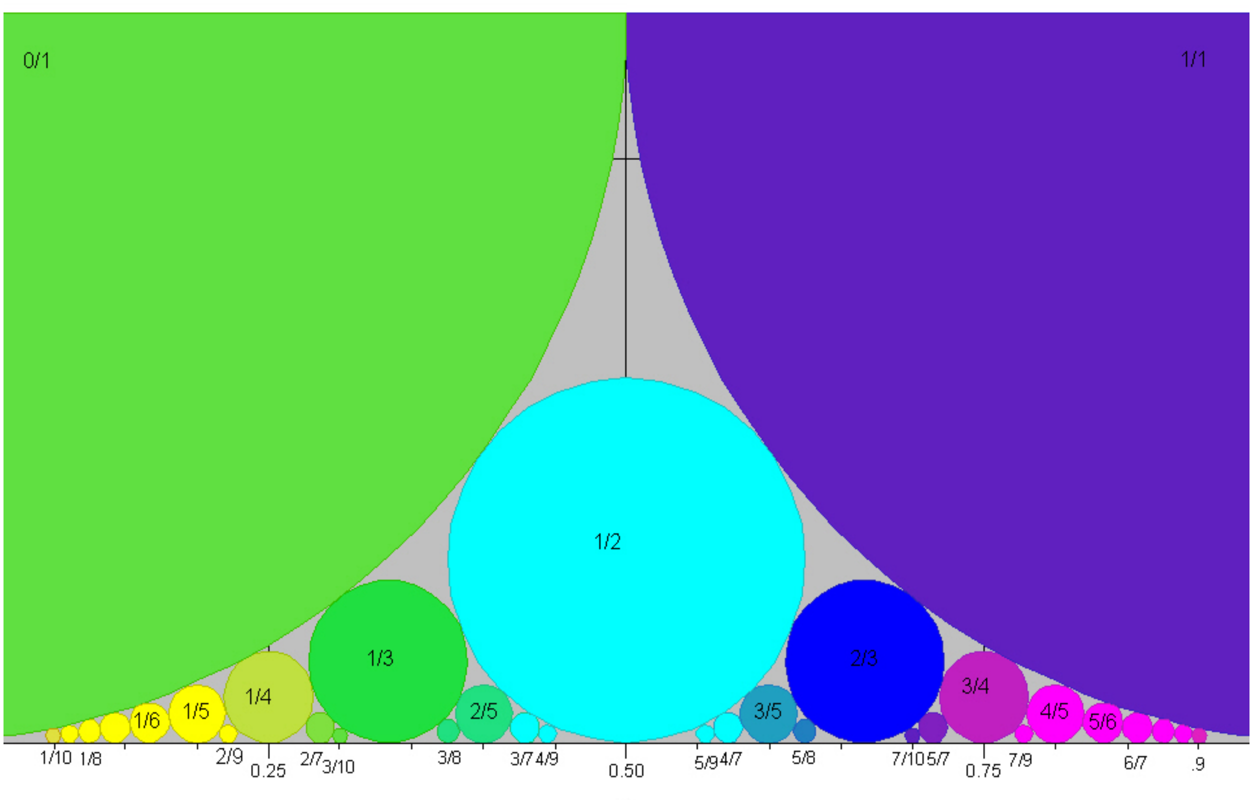
Notice how only gridpoints with coprime coordinates are "visible." Gridpoints such as 2,2 or 2,4 are hidden behind trees growing out of the corresponding gridpoint with the common divisor scaled out. If we flip this drawing vertically, it turns into the graph of the notorious "ruler" function that manages to be continuous at every irrational number, and discontinuous at every rational.



The exact definition is simply 0 at every irrational, and $1/d$ for every rational n/d. Now adjoin points 0 and 1, where the function is 1, and square it:



Finally, let each of these vertical segments be the diameter of a circle:



These are known as the Ford circles. Each circle's x coordinate is the mediant of the x coordinates of the two larger circles that confine it. If this is n/d, then the circle's radius is $1/d^2$.

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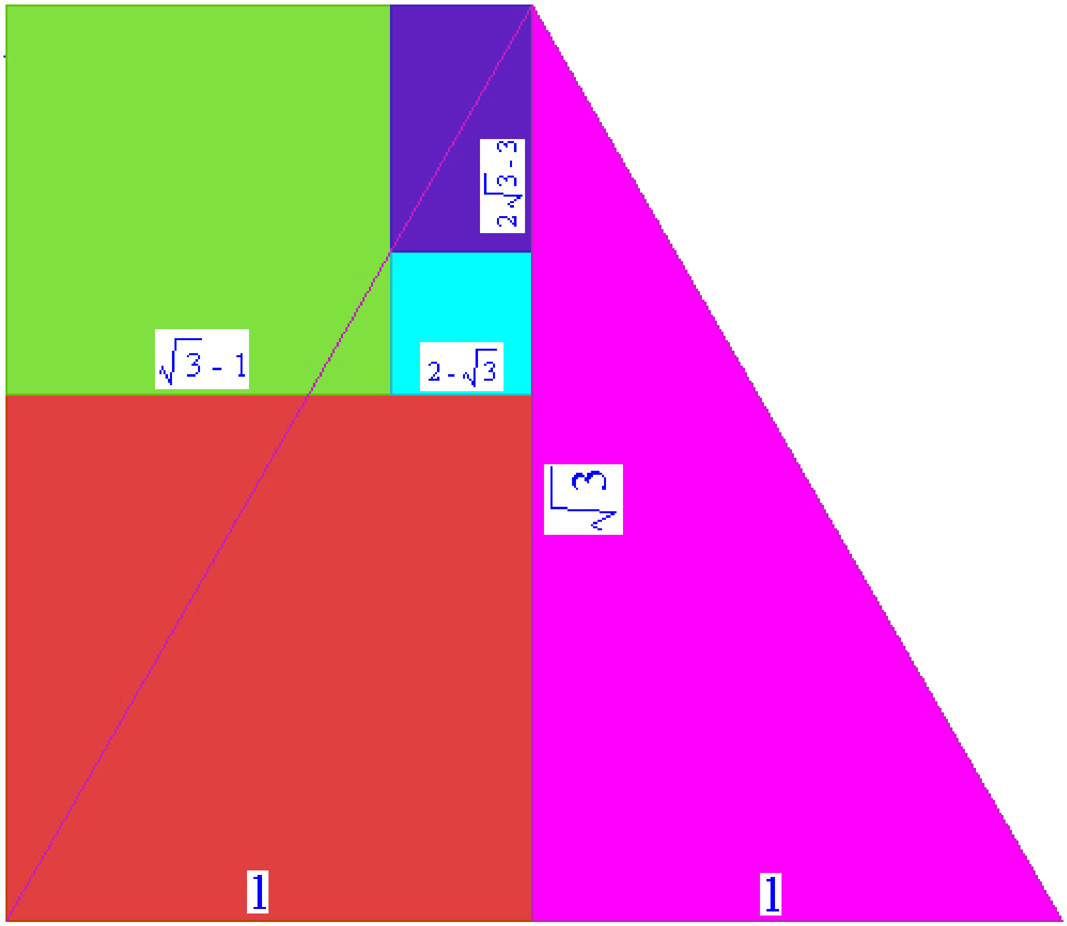
For our last example of rectangle arithmetic, let us investigate whether three gridpoints (*i.e.,* having integer coordinates), can form an equilateral triangle. If we double the size of the triangle (and maybe even if we don't), the midpoint of the base will also be a gridpoint, and we supposedly have a grid rectangle with diagonal slope $= \sqrt{3}$. But $\sqrt{3}$ is irrational, *i.e.*, not a ratio of integers. To see this, try iteratively removing the largest square. As we have seen, when the slope is a ratio of integers, this process terminates (consumes the entire rectangle) in a finite number of steps. The sides are commensurable. But after removing three squares, we have a (dark blue) rectangle



with diagonal slope

$$\frac{2\sqrt{3}-3}{2-\sqrt{3}} = \frac{\left(2-\sqrt{3}\right)\sqrt{3}}{2-\sqrt{3}} = \sqrt{3}.$$

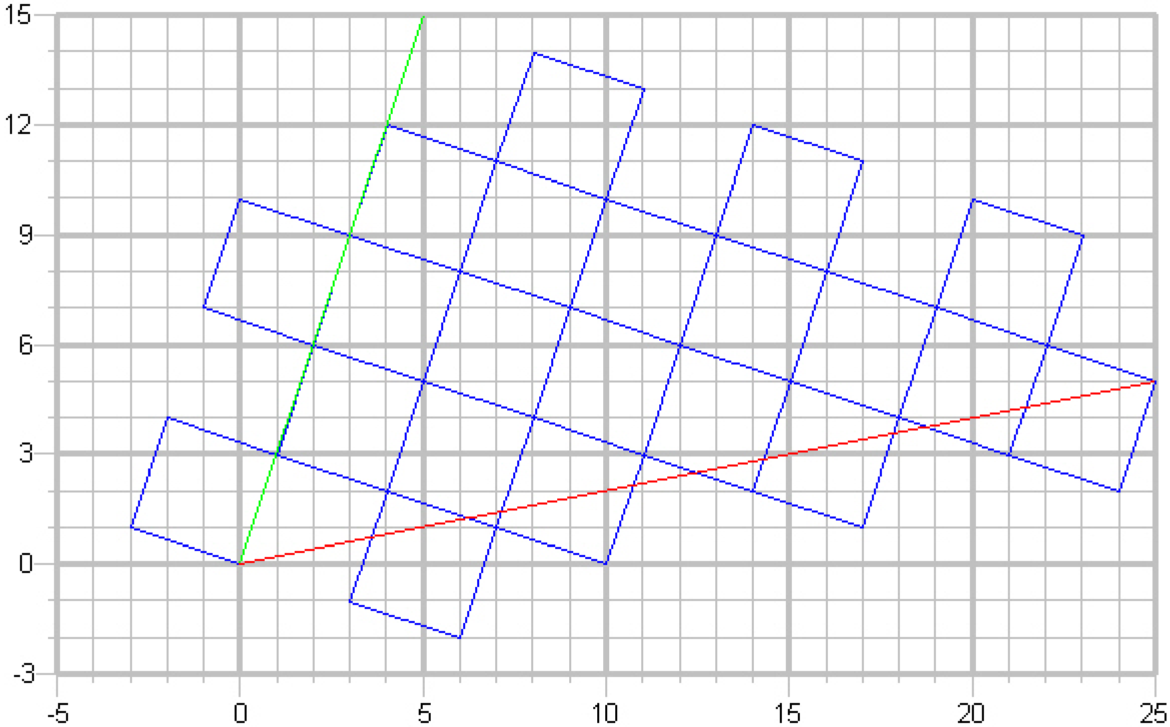
In other words, this rectangle has the same diagonal as the original, so the process will go on forever. In fact this gives the infinite continued fraction for

$$\sqrt{3} = 1 + \cfrac{1}{1+\cfrac{1}{1+\sqrt{3}}} = 1 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{\ddots}}}}}}$$

So in the grid there is no equilateral triangle with a horizontal base. But what about tilting the triangle at some angle? Surprise: The slopes of **all** of the angles in the infinite, two-dimensional grid are rational, and are thus found among the (untilted) grid rectangles. In fact, if two lines through a point have slopes *s* and *t*, the slope of the angle between them is

$$\frac{s-t}{1+s\,t}$$

which is clearly rational when *s* and *t* are. But instead of deriving this formula, there is a more intuitive way to see that there are no new angles to be had by tilting. As an example, we'll use the angle between slopes 3 (green line) and 1/5 (red line), but the argument works in general.



Arbitrarily choosing the slope 3 line (green), make a square grid (blue) using increments (sides of the squares) (1,3) and (3,-1). Now follow the red line from (0,0). The heavier vertical grid lines (gray) interrupt it at (5,1), (10,2), (15,3), &c. Notice that (0,0) is the corner of a blue grid square, and (5,1), (10,2) &c land in various places inside blue squares. But there are at most ten = 3×3 + 1×1 (in this case) different places to land before winding up on another blue corner, in this case, (25,5). But then (0,0), (4,12), and (25,5) form half a rectangle (with a red diagonal) in the blue grid, and by counting blue squares, the angle between the green and red lines has slope 7/4, just as predicted by the formula,

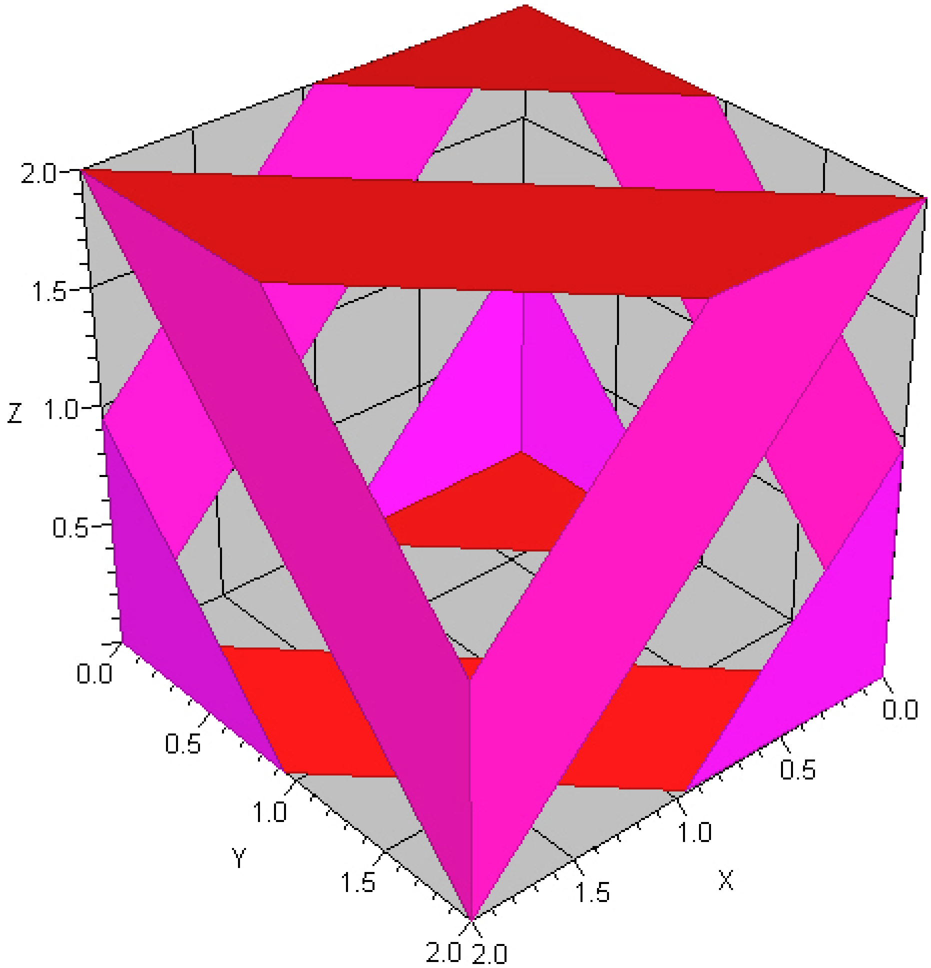
$$\frac{3-\frac{1}{5}}{1+3\,\frac{1}{5}}=\frac{14}{8}=\frac{7}{4}.$$

Notice that the sequence (0,0), (5,1), (10,2),...(25,5) visited exactly half of the ten possible gridpoints in the blue squares. If we simulate wraparound by subtracting multiples of the edges (1,3) and (3,-1) so as to confine the moving point to one square, we have the sequence

(0,0), (1,-1), (2,-2), (3,-3) = (0,-2), (1,-3), (2,-4) = (-1,-3) = (0,0).

Thus for any pair of gridpoints, there is a rectangle of gridpoints with one vertex at the origin, one side through the first point, and one of its diagonals through the second point.

Let us conclude this slope opera by escaping from Flatland while idly asking: Are there equilateral triangles in the **three** dimensional grid? Abundantly! With coordinates limited to just 0, 1, and 2, there are eighty, as well as four regular hexagons:



How many dimensions before we see (regular) octagons, pentagons, dodecagons, ...? We won't! Even in an infinite dimensional grid, the only regular polygons of finite size are triangles, hexagons, and squares. We could probably prescribe the *n*th coordinate of the *k*th vertex of a regular pentagon, say, but infinitely many coordinates would be nonzero, resulting in infinite size. And it would only *approach* regularity as we consider higher and higher dimensions.

Illustrations by Macsyma.

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